Multipartite entanglement of superpositions

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The entanglement of superpositions [Linden *et al.*, Phys. Rev. Lett. **97**, 100502 (2006)]is generalized to the multipartite scenario: an upper bound to the multipartite entanglement of a superposition is given in terms of the entanglement of the superposed states and the superposition coefficients. This bound is proven to be tight for a class of states composed of an arbitrary number of qubits. We also extend the result to a large family of quantifiers, which includes the negativity, the robustness of entanglement, and the best separable approximation measure.

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I. INTRODUCTION

The study of quantum correlations is certainly one of the most challenging issues that physicists have been faced with. Both from an experimental and a theoretical point of view, the characterization of entanglement has proven to be very hard [1]. Even in the simplest scenario, namely, the study of bipartite pure-state entanglement, we still find open questions. Needless to say, the cases of mixed states and multipartite systems are much richer and further from being completely understood.

In recent work [2], Linden, Popescu, and Smolin have raised the following question: Given pure states $|\Psi\rangle$ and $|\Phi\rangle$ on a bipartite system, how is the entanglement of the superposition state

$$|\Gamma\rangle = a|\Psi\rangle + b|\Phi\rangle \tag{1}$$

related to the entanglement of the constituents $|\Psi\rangle$ and $|\Phi\rangle$ and to the coefficients a and b? This apparently simple question was shown to exhibit a rich answer in terms of nontrivial inequalities relating these quantities. In order to quantify the entanglement, the authors of [2] used the von Neuman entropy of the reduced state (often called the entanglement entropy [3]). This is a natural choice, since this quantifier has a clear operational meaning: it gives the number of Bell pairs that can be produced from a large number of copies of an arbitrary entangled state by local operations and classical communication [3]. However, other entanglement quantifiers can also be used and, in fact, distinct bounds for the entanglement of a superposition can be found depending on this choice [5–7].

The main goal of this work is to generalize the ideas raised in [2] to the multipartite scenario. However, instead of working with a specific entanglement quantifier, we have chosen a family of quantifiers called *witnessed entanglement* [8]. This family represents those measures that can be written as

$$E_{\mathcal{W}}(\rho) = \max\{0, -\min_{W\in\mathcal{W}} \mathrm{Tr}(W\rho)\}, \tag{2}$$

where $\mathcal{W}$ is a restricted set of entanglement witnesses [9]. The term "entanglement witness" refers to a Hermitian nonpositive operator that has positive mean value for all separable states; hence a negative mean value indicates the presence of entanglement [9,10]. For an entangled pure state $\rho = |\psi\rangle\langle\psi|$, the witnessed entanglement can be expressed by [11]

$$E_{\mathcal{W}}(\psi) = -\langle\psi|W_{opt}^{\psi}|\psi\rangle, \tag{3}$$

W_{opt}^{ψ} being an optimal witness for the state $|\psi\rangle$ (i.e., a witness satisfying the minimization problem in (2) [12]). This simplified way of writing $E_{\mathcal{W}}$ will be particularly useful to our constructions.

One important fact concerning $E_{\mathcal{W}}$ is that several interesting entanglement quantifiers belong to this class. These quantifiers include concurrence [4], negativity [13,14], robustness of entanglement [15–17], and the best separable approximation [18]. Each one of these examples can be written in the form of Eq. (2) by changing the choice of the set $\mathcal{W}$ [8]. Another advantage of $E_{\mathcal{W}}$ is that it can be directly linked to measurable quantities, since W is a Hermitian operator. Because of that, $E_{\mathcal{W}}$ can be experimentally estimated even for an unknown quantum state [19,20]. It must be stressed, and this is very important in our considered scenario, that $E_{\mathcal{W}}$ can also quantify different kinds of multipartite entanglement: the restricted set $\mathcal{W}$ can be chosen as a set of entanglement witnesses which detect only a certain kind of entangle-

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ment. Besides that, among the witnessed entanglement quantifiers, we can find both operational measures [21–23] (i.e., entanglement quantifiers with some operational meaning) and geometrical ones [18,24,25] (i.e., quantifiers related to geometrical aspects of the state space).

II. MULTIPARTITE ENTANGLEMENT OF SUPERPOSITIONS

The main scope of this work is to obtain an upper bound to the witnessed entanglement of the state (1) based on the entanglement of the superposed states $|\Psi\rangle$ and $|\Phi\rangle$ and the coefficients appearing in the superposition. In this section, we first derive an inequality relating these quantities and then prove its tightness. The witnessed entanglement of $|\Gamma\rangle$ can be written as

$$\begin{aligned}E_{\mathcal{W}}(\Gamma) &= \max\{0,-\min_{W\in\mathcal{W}}\langle\Gamma|W|\Gamma\rangle\}\\ &= \max\{0,-\min_{W\in\mathcal{W}}[|a|^2\langle\Psi|W|\Psi\rangle+|b|^2\langle\Phi|W|\Phi\rangle\\ &\quad+2\mathrm{Re}(a^*b\langle\Psi|W|\Phi\rangle)]\},\end{aligned} \tag{4}$$

an expression that resembles the usual interference pattern originated by superpositions. The minimization problem is solved using the so-called optimal entanglement witness W_{opt} (inside the set $\mathcal{W}$ which defines the quantifier). So we can write

$$\begin{aligned}E_{\mathcal{W}}(\Gamma) &= \max\{0,-|a|^2\langle\Psi|W_{opt}^{\Gamma}|\Psi\rangle-|b|^2\langle\Phi|W_{opt}^{\Gamma}|\Phi\rangle\\ &\quad-2\mathrm{Re}(a^*b\langle\Psi|W_{opt}^{\Gamma}|\Phi\rangle)\}.\end{aligned} \tag{5}$$

Again, W_{opt}^{Γ} denotes a witness that is optimal for the state $|\Gamma\rangle$. Different states usually have different optimal entanglement witnesses. We are naturally led to the inequality

$$\begin{aligned}E_{\mathcal{W}}(\Gamma) &\le \max\{0,-|a|^2\langle\Psi|W_{opt}^{\Psi}|\Psi\rangle\}+\max\{0,\\ &\quad-|b|^2\langle\Phi|W_{opt}^{\Phi}|\Phi\rangle\}+\max\{0,\\ &\quad-2\mathrm{Re}(a^*b\langle\Psi|W_{opt}^{\Gamma}|\Phi\rangle)\}\\ &= |a|^2E_W(\Psi)+|b|^2E_W(\Phi)+2\max\{0,\\ &\quad-\mathrm{Re}(a^*b\langle\Psi|W_{opt}^{\Gamma}|\Phi\rangle)\},\end{aligned} \tag{6}$$

where we have also made use of the inequality $\max\{0,a+b\}\le\max\{0,a\}+\max\{0,b\}$. Attention must now be paid to the interference term. The Cauchy-Schwarz inequality implies

$$E_W(\Gamma)\le|a|^2E_W(\Psi)+|b|^2E_W(\Phi)+2|a||b|\|W_{opt}^{\Gamma}\|. \tag{7}$$

Note that the normalization of the kets involved was used, and we take, as a matter of fact, the norm of an operator as its maximal singular value. Expression (7) relates the entanglement of $|\Gamma\rangle$ to the entanglement of each one of the superposed states (and the coefficients of the superposition) but also depends on the form of the optimal entanglement witness W_{opt}^{Γ}. This dependence on the optimal entanglement witness is expected, as the restrictions in W_{opt}^{Γ} imply the features of the entanglement quantifier we are dealing with.

At this point it is worth asking if inequality (7) can be saturated. Let us choose the negativity as a quantifier for instance. In this case we can compute W_{opt}^{Γ} analytically. For a given state ρ, it is given by the partial transposition of the projector onto the subspace of negative eigenvalues of ρ^{T_A}, where ρ^{T_A} denotes the partial transposition of ρ [26]. It is now easy to see that, for the two-qubit states $|\Phi\rangle=|00\rangle$ and $|\Psi\rangle=|11\rangle$, the inequality (7) becomes $|a||b|\le|a||b|$.

In the previous examples, we used the fact that we knew the optimal witness W_{opt}^{Γ}. Let us now remove this strong assumption. It was shown in Ref. [8] that, if $\mathcal{W}$ [in Eq. (2)] is the set of entanglement witnesses satisfying $-nI\le W\le mI$, where $m,n\ge0$, $E_{\mathcal{W}}$ is an entanglement monotone [27]. Setting $k=\max(m,n)$ we have

$$E_{\mathcal{W}}(\Gamma)\le|a|^2E_{\mathcal{W}}(\Psi)+|b|^2E_{\mathcal{W}}(\Phi)+2k|a||b|. \tag{8}$$

As our main goal here is to work in the multipartite case, it would be interesting to find examples of multipartite states for which relation (8) is saturated. The main barrier to be overcome in this case is the fact that it is not known, in general, how to compute multipartite entanglement quantifiers. Nevertheless, we develop a way of calculating the generalized robustness of entanglement for Greenberger-Horne-Zeilinger– (GHZ-)like states and use this information to prove the tightness of inequality (8) regardless of the number of particles involved.

The generalized robustness of entanglement [17] admits two representations, one in terms of how robust the entanglement of a state is against arbitrary noise and the other as a witnessed entanglement. Let us present both definitions precisely.

Definition 1. The generalized robustness of entanglement of a state ρ is given by

$$R_g(\rho)=\inf_{\pi\in\mathcal{D}}\min\{s:\sigma(\rho,\pi,s)\in\mathcal{S}\}, \tag{9}$$

where σ denotes the state

$$\sigma(\rho,\pi,s)=\frac{\rho+s\pi}{1+s}, \tag{10}$$

$\mathcal{D}$ the set of all density operators, and $\mathcal{S}$ the set of separable ones (with respect to the specific form of entanglement that is considered).

Definition 2. $R_g(\rho)$ is the witnessed entanglement $E_{\mathcal{W}}(\rho)$ when $\mathcal{W}$ is the set of witness operators satisfying $W\le I$.

The equivalence of these definitions was proven in [8]. We make use of both to show that for the N-qubit family of states

$$|\mathrm{GHZ}_N(\phi)\rangle=\frac{|0^{\otimes N}\rangle+e^{i\phi}|1^{\otimes N}\rangle}{\sqrt{2}}, \tag{11}$$

the inequality (8) is saturated. Clearly, if one chooses an arbitrary state π such that the state $\sigma(\rho,\pi,s)$ is separable for some value of s, this number s gives an upper bound for the value of $R_g(\rho)$. On the other hand, taking an arbitrary entanglement witness W for the state ρ satisfying the condition $W\le I$, $-\mathrm{Tr}(W\rho)$ gives a lower bound to $R_g(\rho)$ according to (2). We will now establish lower and upper bounds for $R_g(\mathrm{GHZ}_N(\phi))$ that turn out to be equal, getting the exact value of this quantity and also the value of k needed for the bound (8).

Upper bound. Consider, in Eq. (10),

$$\rho = |\mathrm{GHZ}_N(\phi)\rangle\langle \mathrm{GHZ}_N(\phi)| \tag{12}$$

and

$$\pi = |\mathrm{GHZ}_N(\phi)_\perp\rangle\langle \mathrm{GHZ}_N(\phi)_\perp|, \tag{13}$$

where

$$|\mathrm{GHZ}_N(\phi)_\perp\rangle = \frac{|0^{\otimes N}\rangle - e^{i\phi}|1^{\otimes N}\rangle}{\sqrt{2}}. \tag{14}$$

Using the Peres criterion [28] we see that σ has positive partial transposition only for $s=1$. Moreover, for this point it can be directly verified that σ is also separable. So we get

$$R_g(\mathrm{GHZ}_N(\phi)) \leq 1. \tag{15}$$

Lower bound. The following operator is a genuine entanglement witness for the state $|\mathrm{GHZ}_N(\phi)\rangle$ [25,29]:

$$W = I - 2|\mathrm{GHZ}_N(\phi)\rangle\langle \mathrm{GHZ}_N(\phi)|, \tag{16}$$

which clearly satisfies the condition $W \leq I$. Hence, definition (2) leads to

$$-\mathrm{Tr}[W|\mathrm{GHZ}_N(\phi)\rangle\langle \mathrm{GHZ}_N(\phi)|] = 1 \leq R_g(\mathrm{GHZ}_N(\phi)). \tag{17}$$

As the upper bound (15) and lower bound (17) coincide, we have that $R_g(\mathrm{GHZ}_N(\phi))=1$, and can also conclude that the witness (16) satisfies the minimization problem in (2). It then allows us to extract the value $k=1$.

Putting all these facts together, we conclude that the inequality (8) saturates for the class of states (11).

III. CONCLUSIONS

We extended the notion of entanglement of superpositions to the multipartite scenario. An inequality relating the entanglement of quantum states to the entanglement of the state constructed through their superposition was found. This inequality was proven to be tight for a family of N-qubit states and a choice of entanglement quantifier. Moreover, a large class of entanglement quantifiers, with both operational and geometrical meanings, was put in this context.

It is also worth noting that the inequalities derived here can be extended for the case where more than two states are superposed [30]. Future research could include the study of other examples of states and quantifiers treated in our general perspective.

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