Quantum Nonlocality and Partial Transposition for Continuous-Variable Systems

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A continuous-variable Bell inequality, valid for an arbitrary number of observers measuring observables with an arbitrary number of outcomes, was recently introduced [Cavalcanti *et al.*, Phys. Rev. Lett. **99**, 210405 (2007)]. We prove that any n-mode quantum state violating this inequality with quadrature measurements necessarily has a negative partial transposition. Our results thus establish the first link between nonlocality and bound entanglement for continuous-variable systems.

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During most of the history of quantum mechanics, the concepts of entanglement and nonlocality were considered as a single feature of the theory. It was based on the discussion of nonlocality started by Einstein, Podolsky, and Rosen [1] that Schrödinger stressed the importance of entanglement in the understanding of quantum systems [2]. Later, Bell derived experimentally testable conditions, known as Bell inequalities, to verify the nonlocal character of entanglement [3].

It was only with the recent advent of quantum information science that the relation between these two concepts started to be considered in depth. On one hand, we need entanglement for a state to be nonlocal, where by nonlocality we understand incompatibility with local-hidden-variable (LHV) models [4]. But, on the other hand, we know that some entangled states admit a LHV model [5]. The situation is even richer due to the fact that there exist other meaningful scenarios where sequences of measurements [6] or activationlike processes [7] allow detecting the hidden nonlocality of quantum states. More in general, the relation between these concepts is still not fully understood. Clarifying this relation is highly desirable, for it would lead us to ultimately grasp the very essence of quantum mechanics.

One way to tackle this problem is by studying the relation between nonlocality and other concepts regularly related to entanglement. In this direction, Peres conjectured [8] that any state having a positive partial transposition (PPT) should admit a LHV model, or, equivalently, any state violating a Bell inequality should have a negative partial transposition (NPT). Partial transposition has been proven one of the most useful tools in the study of entanglement. As shown by Peres [9], any NPT state is necessarily entangled. However, positivity of the partial transposition (PPT) represents a necessary, but not sufficient, condition for a state to be separable. Indeed, partial transposition is just the simplest example of positive maps, linear maps that are useful for the detection of mixed-state entanglement [10]. A second fundamental result on the connection between partial transposition and entanglement was to notice that all PPT states are nondistillable [11]. In other words, if an entangled state is PPT, it is impossible to extract pure-state entanglement out of it by local operations assisted by classical communication, even if the parties are allowed to perform operations on many copies of the state. In this way, PPT entanglement was regarded, for a long time, as a useless kind of quantum correlation [12].

Proving (or disproving) Peres' conjecture in full generality represents one of the open challenges in quantum information theory. The proof of the conjecture has up to now been achieved only for some particular cases; if we label the nonlocality scenario as is customary by the numbers (n, m, o), meaning that n parties can choose between m measurement settings of o outcomes each, the most general proof obtained so far is for correlation Bell inequalities in the $(n, 2, 2)$ case [13,14]. To our knowledge, there is no result for a larger number of settings and/or outcomes and, in particular, for the relevant case of continuous-variable (CV) systems. These systems are very promising for loophole-free Bell violations, due to the high efficiency achieved in homodyne detection [15].

Unfortunately, there have been few results so far on Bell inequalities for CV systems [16]. Recently, Cavalcanti, Foster, Reid and Drummond (CFRD) introduced a very general Bell inequality for the $(n, 2, o)$ scenario with arbitrary n and o, which works in particular when $o \to \infty$, the CV case. We make use of the Shchukin and Vogel (SV) NPT criterion [17] to show that the CFRD inequality with two arbitrary quadratures as settings on each site is not violated for multipartite PPT states, thus proving Peres' conjecture in this relevant scenario. This is the first result on the connection between partial transposition and Bell inequalities for CV systems, and takes us a step further in understanding the relation between entanglement and nonlocality. After proving this result, we discuss the practical applicability of the CFRD inequality, showing that no two-mode quantum state can violate it when performing two homodyne measurements on each site.

CFRD Bell inequality.—In Ref. [18], the authors use the fact that the variance of any function of random variables must necessarily be positive to get general Bell inequalities. By choosing functions of local observables one can find discrepancies between the quantum and the classical predictions just using the fact that in the quantum case these observables are given by Hermitian operators (usually satisfying nontrivial commutation relations), while in an LHV scenario the observables are just numbers, given *a priori* by the hidden variables (and obviously commuting with each other). Interestingly, this idea can lead to strong Bell inequalities as the well-known Mermin, Ardehali, Belinskii and Klyshko inequality [19]. More importantly for the present discussion, the CFRD approach works for unbounded observables as well, leading to Bell inequalities for CV systems.

Consider a complex function C_n of the local real observables $\{X_k, Y_k\}$, where k labels the different parties, defined as:

$$C_n = \tilde{X}_n + i\tilde{Y}_n = \prod_{k=1}^{n}(X_k + iY_k). \tag{1}$$

Applying the positivity of the variance of both its real ($\tilde{X}_n$) and imaginary ($\tilde{Y}_n$) part, and assuming LHV (i.e., setting commutators to zero) we obtain [18]:

$$\langle\tilde{X}_n\rangle^2 + \langle\tilde{Y}_n\rangle^2 \leq \left\langle\prod_{k=1}^{n}(X_k^2 + Y_k^2)\right\rangle. \tag{2}$$

This inequality must be satisfied by LHV models for any set of observables $\{X_k, Y_k\}$, regardless of their spectrum. In particular, it applies to CV observables.

Consider, for each site, two arbitrary quadratures defined in terms of the annihilation (creation) operators $\hat{a}_k$ ($\hat{a}_k^\dagger$) as:

$$\begin{aligned}\hat{X}_k &= \hat{a}_k e^{-i\theta_k} + \hat{a}_k^\dagger e^{i\theta_k},\\ \hat{Y}_k &= \hat{a}_k e^{-i(\theta_k+\delta_k+s_k\pi/2)} + \hat{a}_k^\dagger e^{i(\theta_k+\delta_k+s_k\pi/2)},\end{aligned} \tag{3}$$

where $-\pi/2 < \delta_k < \pi/2$ quantifies the departure from orthogonality, $s_k = \pm 1$, and $[a_k, a_k^\dagger] = 1$. With these parameters all possible choices of angles are covered, noting that $\delta_k = -\pi/2, \pi/2$ corresponds to measuring only one quadrature. Note that the measurement scenario used in Ref. [18] is a particular case of (3) corresponding to $\delta_k = 0$. We define new operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ as:

$$\begin{aligned}\hat{b}_k &= \frac{(\hat{X}_k + e^{is_k\pi/2}\hat{Y}_k)e^{i\theta_k}}{2\sqrt{\cos\delta_k}};\\ \hat{b}_k^\dagger &= \frac{(\hat{X}_k + e^{-is_k\pi/2}\hat{Y}_k)e^{-i\theta_k}}{2\sqrt{\cos\delta_k}}.\end{aligned} \tag{4}$$

These operators satisfy the usual commutation relation $[\hat{b}_k, \hat{b}_k^\dagger] = 1$, independently of the s_k. Inverting these equations we get:

$$\begin{aligned}\hat{X}_k &= \sqrt{\cos\delta_k}(\hat{b}_k e^{-i\theta_k} + \hat{b}_k^\dagger e^{i\theta_k}),\\ \hat{Y}_k &= \sqrt{\cos\delta_k}(\hat{b}_k e^{-i(\theta_k+s_k\pi/2)} + \hat{b}_k^\dagger e^{i(\theta_k+s_k\pi/2)}).\end{aligned} \tag{5}$$

Plugging these operators in (2) we arrive at the CFRD inequality for *arbitrary* quadratures:

$$\left|\left\langle\prod_k \hat{B}_k(s_k)\right\rangle\right|^2 \leq \frac{1}{\prod_k \cos\delta_k}\left\langle\prod_k\left(\cos\delta_k \hat{b}_k^\dagger\hat{b}_k + \frac{1}{2}\right)\right\rangle, \tag{6}$$

with $\hat{B}_k(1) = \hat{b}_k$ and $\hat{B}_k(-1) = \hat{b}_k^\dagger$. In what follows, we show that all states violating this inequality must be NPT according to some bipartition. In order to prove this, we need to recall Shchukin and Vogel's (SV) criterion [17].

SV criterion.—A necessary and sufficient condition for the positivity of the partial transposition of a CV state, given in terms of matrices of moments, was introduced and further generalized to the multipartite case in Ref. [17]. When dealing with many parties, one must analyze the positivity of the partial transposition for the different partitions of the system into two groups. We say that a state is PPT when it is PPT according to *all* bipartitions. Let us introduce the SV criterion for the multipartite scenario.

When considering the partial transposition of a quantum state ρ with respect to a given bipartition of the system, we label by I the set of parties that we choose to transpose, which also defines the corresponding bipartition. We construct a matrix of moments M^I whose elements are given by the expectation values:

$$M_{st}^I = \left\langle\prod_{i\in I}\hat{b}_i^{\dagger q_i}\hat{b}_i^{p_i}\hat{b}_i^{\dagger k_i}\hat{b}_i^{l_i}\prod_{i\in\bar{I}}\hat{b}_i^{\dagger l_i}\hat{b}_i^{k_i}\hat{b}_i^{\dagger p_i}\hat{b}_i^{q_i}\right\rangle, \tag{7}$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ and $\mathbf{l} = (l_1, \ldots, l_n)$ correspond to row index s, and $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ correspond to column index t, with some prescribed ordering that is not relevant for our purposes (see [17] for details), and $\bar{I}$ denotes the complement of I, that is, those parties which we choose *not* to transpose. We stress that, for fixed row and column indices, the ordering of the operators entering the corresponding matrix element will depend on the bipartition I.

Shchukin and Vogel's criterion says that, for a state to be PPT according to bipartition I, all principal minors of M^I should be nonnegative [20]. So, for a state to be PPT according to *all* bipartitions, all principal minors of *all* matrices M^I must be nonnegative for all nontrivial bipartitions I. By nontrivial bipartitions we mean that we exclude the bipartition labeled by $I = \varnothing$, as well as that labeled by $I = \mathcal{N}$, the entire set, both corresponding to no transposition at all. In these cases, the criterion speaks about the positivity of the state itself, instead of its partial transposition.

Nonlocality implies NPT.—We are now in the position of proving that any state violating the generalized CFRD inequality (6) is necessarily NPT. We begin by expanding the products in the right-hand side (RHS) of inequality (6) as follows:

$$\frac{1}{\prod_k \cos\delta_k}\left\langle\prod_k\left(\cos\delta_k\hat{N}_k+\frac{1}{2}\right)\right\rangle=\frac{1}{\prod_k \cos\delta_k}\left(\frac{1}{2^n}+\frac{1}{2^{n-1}}\sum_{i_1=1}^{n}\cos\delta_{i_1}\langle\hat{N}_{i_1}\rangle+\frac{1}{2^{n-2}}\sum_{i_1=1}^{n}\sum_{i_2>i_1}^{n}\cos\delta_{i_1}\cos\delta_{i_2}\langle\hat{N}_{i_1}\hat{N}_{i_2}\rangle+\ldots\right.$$
$$\left.+\frac{1}{2}\sum_{i_1=1}^{n}\sum_{i_2>i_1}^{n}\cdots\sum_{i_{n-1}>i_{n-2}}^{n}\cos\delta_{i_1}\cos\delta_{i_2}\cdots\cos\delta_{i_{n-1}}\langle\hat{N}_{i_1}\hat{N}_{i_2}\cdots\hat{N}_{i_{n-1}}\rangle\right)+\left\langle\prod_k\hat{N}_k\right\rangle, \quad (8)$$

where we made use of the number operators defined as $\hat{N}_k \equiv \hat{b}_k^\dagger \hat{b}_k$. We now take all but the last term on the RHS of Eq. (8) and call their sum S^2, so that:

$$\frac{1}{\prod_k \cos\delta_k}\left\langle\prod_k\left(\cos\delta_k\hat{N}_k+\frac{1}{2}\right)\right\rangle = S^2+\left\langle\prod_k\hat{N}_k\right\rangle. \quad (9)$$

Note that S^2 is a nonnegative quantity, since $\pi/2 < \delta_k < \pi/2$ and the expectation value of a product of number operators is always nonnegative. We can rewrite inequality (6) as:

$$\left\langle\prod_k\hat{N}_k\right\rangle-\left\langle\prod_k\hat{B}_k(s_k)\right\rangle\left\langle\prod_k\hat{B}_k(-s_k)\right\rangle \geq -S^2. \quad (10)$$

The key point in the proof is to realize that, for any choice of the parameters s_k, the left-hand side (LHS) of Eq. (10) is just one of the principal minors of M^I, provided we choose the bipartition I appropriately. The principal minor we will look at is:

$$D^I = \begin{vmatrix} 1 & \langle\prod_k \hat{B}_k(s_k)\rangle \\ \langle\prod_k \hat{B}_k(-s_k)\rangle & \eta_I \end{vmatrix}, \quad (11)$$

where η_I depends on the bipartition I, and which we want to take the form $\eta_I = \langle\prod_k \hat{N}_k\rangle$.

Looking at the elements of the matrix of moments M^I given by Eq. (7), we note that the indices labeling the diagonal element that has one creation operator $\hat{b}_k^\dagger$ and one annihilation operator $\hat{b}_k$ in normal order are $l_k = 1$, $k_k = 0$, $p_k = 0$, and $q_k = 1$. The corresponding upper right element is in turn labeled, for the k part, by $l_k = 0$, $k_k = 0$, $p_k = 0$, and $q_k = 1$. If we have the choice of setting $s_k = -1$, we want this to correspond to a creation operator $\hat{b}_k^\dagger$ appearing in this position, which means that our bipartition must be such that I includes site k. Conversely, if we have, for a different k, $s_k = 1$, site k should *not* be in I.

Hence, if we choose the bipartition as that labeled by I including all sites with setting $s_k = -1$, we get $\eta_I = \langle\prod_k \hat{N}_k\rangle$, and thus:

$$D^I = \left\langle\prod_k\hat{N}_k\right\rangle-\left\langle\prod_k\hat{B}_k(s_k)\right\rangle\left\langle\prod_k\hat{B}_k(-s_k)\right\rangle. \quad (12)$$

It follows that a violation of inequality (10) implies that $D^I < 0$, and the violating state must be NPT according to bipartition I, or just NPT, which concludes the proof.

We note that if all s_k are equal to either 1 or -1, this corresponds, respectively, to $I = \varnothing$ or $I = \mathcal{N}$, meaning no transposition at all. As we mentioned above, in this case the positivity of the minors speaks no longer about the positivity of the partial transpose of the state but about the positivity of the state itself. A violation for this choice of parameters, thus, would mean that the state is not positive semidefinite, which is unphysical.

Applicability of the CFRD inequality.—Before concluding, we would like to discuss the applicability of the CFRD inequality. In particular, we now show that in the case of two parties, the CFRD inequality is never violated for measurements on two quadratures per site. An example of violation of this inequality corresponding to measurements on orthogonal quadratures applied to a ten-mode catlike state was given in the original reference [18]. There, it was also shown that the quantum violation of the inequality increases exponentially with the number of modes [18]. However, in spite of its elegance and conceptual relevance, at present there is no feasible scheme [21] producing a violation of the CFRD inequality. This remains an interesting open question.

Let us start by considering systems of two parties with measurements on arbitrary quadratures. Applying the positivity of the variance for the real and imaginary parts of C_2 [see (1)] without neglecting the terms containing commutators we get:

$$\underbrace{\langle\tilde{X}_2\rangle^2+\langle\tilde{Y}_2\rangle^2-\left\langle\prod_{k=1}^{2}(X_k^2+Y_k^2)\right\rangle}_{\beta_2} \leq -\langle[X_1,Y_1][X_2,Y_2]\rangle. \quad (13)$$

The Bell inequality (2) follows by setting the right-hand side (RHS) of this inequality to zero, and we are left with $\beta_2 \leq 0$, since for LHV models all commutators are null. So, in order to have a violation we need to find a state such that $\beta_2 > 0$. We are going to show that this never happens with the choice (3). Indeed, taking the settings of (3), the RHS of (13) becomes $4s_1s_2\cos\delta_1\cos\delta_2$, so we have $\beta_2 \leq 4s_1s_2\cos\delta_1\cos\delta_2$. If the parameters are chosen to be different, i.e., $s_1 = -s_2 = \pm 1$, we have that $\beta_2 \leq -4\cos\delta_1\cos\delta_2 < 0$ for all quantum states, and then there is no violation in this case. As we have previously discussed for arbitrary n, no violation can take place for the case in which all the s_k parameters are equal, $s_1 = s_2$.

Concluding remarks.—Despite years of effort, little is known about which entangled states admit a LHV model, that is, can be simulated using classical correlations. Peres' conjecture represents one of the most interesting open problems related to this fundamental question. Our results are the first to provide support for its validity in the CV regime. It is also the first result beyond the two-setting two-outcome scenario, since all previous partial proofs of the conjecture were for the case of two measurements of two outcomes per site. This gives more support to the belief that the impossibility of distilling entanglement is intimately linked to the existence of an LHV model for a given quantum state.

Finally, CV Bell inequalities suitable for practical tests are very desired due to the high control attained in CV photonic experiments that will allow a loophole-free demonstration. We have shown here that the CFRD can never be violated by two-mode states with quadrature measurements. It is then a relevant open question to construct Bell inequalities suitable for CV systems that can be violated by states consisting of a small number of modes. Future research in this direction could, for instance, involve the study of CV Bell inequalities involving more measurements per site [22].

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