

Frustration, Area Law, and Interference in Quantum Spin Models

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(Received 28 June 2007; revised manuscript received 2 June 2008; published 29 October 2008)

We study frustrated quantum systems from a quantum information perspective. Within this approach, we find that highly frustrated systems do not follow any general “area law” of block entanglement, while weakly frustrated ones have area laws similar to those of nonfrustrated systems away from criticality. To calculate the block entanglement in systems with degenerate ground states, typical in frustrated systems, we define a “cooling” procedure of the ground state manifold and propose a frustration degree and a method to quantify constructive and destructive interference effects of entanglement.

DOI: 10.1103/PhysRevLett.101.187202

PACS numbers: 75.10.Jm, 03.67.–a, 75.50.Ee

The study of entanglement in spin lattice models has provided novel insight into the complexity of magnetic ordering and quantum phase transitions in a large variety of models [1]. In particular, in bipartite partitions of certain many-body systems, the block entanglement (BE) evaluated in the (gapped) ground state (GS) of the system is proportional to the area of the block boundary. More complex relations arise at criticality (nongapped systems). These generic relations between entanglement and area are known under the name of *area laws* [2].

There are several intriguing questions concerning these relations [3], with respect to systems with unique ground states. Within this context, one may ask whether the scaling of entanglement can also help to characterize frustrated many-body systems [4]. Typically, frustrated systems exhibit a large GS degeneracy, or quasidegeneracy, and a rich phase diagram [5]. Frustration in spin systems arises when the GS spin configuration does not simultaneously satisfy all the constraints imposed by the Hamiltonian; it may be caused by the lattice geometry [as in Ising antiferromagnets (AFM) on a triangular lattice] or by the presence of disorder (as in spin glasses). Recently, the relation of frustration to high- T_c superconductivity, as well as the discovery of “exotic” frustrated phases, has triggered a renewed interest in the subject [5–7]. Ultracold atomic gases offer unprecedented ways to control many-body systems, and provide a perfect playground to study disordered and frustrated systems. In particular, this should allow for experimental study of effects discussed in this Letter (cf. [8]; see [9] for a review).

The *main thesis* of this Letter is that highly frustrated systems do not follow any general area law, while weakly frustrated systems have area laws similar to those of nonfrustrated (NF) systems (away from criticality). We introduce three tools to characterize frustrated systems and deal with the above thesis: (i) a paradigm of a certain “cooled state” for finding area laws for systems which have a degenerate ground state manifold, including frustrated sys-

tems, (ii) a “frustration degree” for quantifying frustration in a system, and (iii) a method for quantifying constructive and destructive interference of entanglement. Using these tools, we prove the above thesis for six prototype frustrated spin models, each having a different type of frustration. More specifically, we show that for the Ising model with long-range (LR) AFM interactions (1), and the LR AFM Heisenberg model (2), with frustration degrees close to unity, there is a complete departure from the usual area law for NF systems. However, when we consider systems whose frustration degree is much smaller, we find area laws similar to those of NF systems (away from criticality); i.e., the entropy of a block of characteristic length L , in a system of N particles in dimension D , scales as L^{D-1} , when $L^D \ll N$. Among this class, we have analyzed a two-dimensional (2D) $J_1 - J_2 - J_3$ Heisenberg model with resonating valence bond (RVB) states as GSs (3), the Shastry-Sutherland model (in 2D) (4) [10], the one-dimensional (1D) $J_1 - J_2$ Heisenberg model at the Majumdar-Ghosh (MG) point (5) [11], and finally an Ising chain with a single disordered interaction (6).

Cooling procedure.—To treat the problem of degeneracy of the GS manifold, we introduce a “cooling” method. For a system of several subsystems (e.g., spins), we choose an initial *product* state $|\Psi_0\rangle = \prod_i |\psi_i\rangle$ (i runs over all subsystems) and “cool” $|\Psi_0\rangle$ to below a desired energy level \mathbb{E} , by projecting $|\Psi_0\rangle$ onto the subspace spanned by all energy eigenstates with energies \mathbb{E} . This “quenching” method is a caricature of evaporative cooling of trapped atomic gases, where atoms having energy above a certain value are removed from the trap. So, the resulting state is

$$|\Phi_{\mathbb{E}}\rangle = (1/\sqrt{Z})P[\leq \mathbb{E}]|\Psi_0\rangle, \quad (1)$$

where $P[\leq \mathbb{E}] = \sum_{\mathbb{E}_i \leq \mathbb{E}} P_i[\mathbb{E}_i]$, with $P_i[\mathbb{E}_i]$ being the projector onto the eigenstates of energy \mathbb{E}_i , and $Z = \langle \Psi_0 | P[\leq \mathbb{E}] | \Psi_0 \rangle$. For an N -spin state $|\Phi_{\mathbb{E}}\rangle$, we characterize the entanglement between k spins and the rest of the system by the von Neumann entropy

$E_{k:N-k} = S(\text{tr}_k |\Phi_{\mathbb{E}}\rangle\langle\Phi_{\mathbb{E}}|)$ —the unique asymptotic measure of entanglement for pure states in bipartite splittings [12]. We study the scaling of BE $E_{k:N-k}$, for large N , where the initial state $|\Psi_0\rangle$ is chosen to maximize $E_{N/2:N/2}$ [13].

Frustration degree.—Before proceeding further, we define a *frustration degree* (\mathcal{F}) for quantum spin models. Frustration is a classical concept, and thus connected to the classical configurations of the quantum system. Given a Hamiltonian \mathcal{H} , and a corresponding ground state $|\mathcal{G}\rangle$, we replace the one-body, two-body, etc. terms in \mathcal{H} by the corresponding Ising ones (i.e., $\sigma_i^z \sigma_j^z$, etc.) to obtain \mathcal{H}^I , discarding any constant term. Here σ^α ($\alpha = x, y, z$) are the Pauli operators, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. We find the terms \mathcal{H}_f^k (\mathcal{H}_{nf}^l) of \mathcal{H}^I that give rise to positive (negative or zero) energies in $|\mathcal{G}\rangle$. For a given $|\mathcal{G}\rangle$, $\mathcal{H}^I = \sum_k \mathcal{H}_f^k + \sum_l \mathcal{H}_{nf}^l$. We define the frustration degree of the system described by \mathcal{H} as $\mathcal{F} = \text{avg} \frac{\sum_{|\mathcal{G}\rangle} \langle \mathcal{H}_f^k | \mathcal{G} \rangle}{\sum_{|\mathcal{G}\rangle} \langle \mathcal{H}^I | \mathcal{G} \rangle}$, where the average (avg) is taken over all ground states of \mathcal{H} . Note that the denominator cannot vanish. In the case of isotropic Heisenberg ($\vec{\sigma}_i \cdot \vec{\sigma}_j$) and XY ($\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y$) interactions, to find \mathcal{F} , one may replace $\vec{\sigma}_i$ and (σ_i^x, σ_i^y) by SU(2) and SO(2) symmetric classical vectors, of appropriate lengths, respectively. With these concepts at hand, we move to study the different cases.

Case 1: Ising model with AFM LR interactions: Ising gas.—The governing Hamiltonian, for $2m$ spins, is $H_{\text{LR}}^{\text{Ising}}(\lambda) = (J/2m)(S - 2m\lambda)^2 \equiv (J/2m) \times (2 \sum_{i,j=1}^{2m} \sigma_i^z \sigma_j^z - 4m\lambda \sum_{i=1}^{2m} \sigma_i^z + \text{const})$, where $S = \sum_i \sigma_i^z$, $0 \leq \lambda \leq 1$. In the frustrated case ($J > 0$), $\mathcal{F} = (1 + 2\lambda - \lambda^2 - 1/m)/(1 + \lambda)^2$. For large systems, with $\lambda = 0$, $\mathcal{F} \approx 1$. For the initial state $|\Psi_0^I\rangle = \prod_i (\alpha|0\rangle + \beta|1\rangle)_i$, with $\alpha\beta \neq 0$ (with $|0\rangle$ and $|1\rangle$ being the eigenstates of σ^z with eigenvalues ± 1 , respectively), the cooled state (which is independent of α, β) is $|\Phi_{\text{LR}}^{\text{Ising}}(\lambda)\rangle =$ superposition of all states with $m(1 + \lambda)|0\rangle$'s and $m(1 - \lambda)|1\rangle$'s, where $2m\lambda$ is a positive integer $\leq 2m$. For arbitrary $2m$ and magnetization λ , and for any bipartite splitting $k : 2m - k$, the reduced density matrix can be found analytically: for even $k < m(1 - \lambda)$, $\rho_k = \sum_{i=0}^k [k, i][2m - k, m\mu - i] |W_{k-i}\rangle\langle W_{k-i}| / [2m, m\mu]$, where $\mu = \lambda + 1$, $[a, b] = \frac{a!}{(a-b)!b!}$, and $|W_j\rangle$ denotes the normalized equal superposition state of $j|1\rangle$'s and rest $|0\rangle$'s, i.e., the (generalized) “W” state. Any other partition can be found similarly. For a given k , $E_{k:2m-k}$ increases (and converges to a constant) with $2m$. For a fixed m , using Stirling approximation, $E_{k:2m-k} \approx -\sum_{i=0}^k e_i \log_2 e_i$, with $e_i = \frac{1}{2^k} (1 + \lambda)^i (1 - \lambda)^{k-i} [k, i]$, for large m and k , such that $k \ll m$. In the same limit, $E_{k:2m-k} \approx \frac{1}{2} \log_2 [(1 - \lambda^2)k]$. The leading term to the scaling is $\frac{1}{2} \log_2 k$, independently of λ —a logarithmic divergence of BE for all frustrated AFM LR Ising gases. Since this system is effectively in “infinite” dimensions, there is a clear departure from any “area law” (which could be $k^{1-1/D}$ with $D \rightarrow \infty$).

Similar divergence is also obtained even if the quenching is performed over a few low-lying energy manifolds. In contrast, for the ferromagnetic ($J < 0$) NF LR Ising model (say, at $\lambda = 0$), the cooled state $(|0^{*2m}\rangle + |1^{*2m}\rangle)/\sqrt{2}$ has $E_{k:2m-k} = 1$, for all $k, 2m$.

Case 2: Heisenberg model with AFM LR interactions: Heisenberg gas.—The Hamiltonian, for $2m$ spins, is $H_{\text{LR}}^{\mathcal{H}} = (J/2m) \sum_{i,j=1}^{2m} \vec{\sigma}_i \cdot \vec{\sigma}_j$ ($J > 0$). The ground state manifold can be defined by labeling the $2m$ spins as “black” (call them \mathcal{B}) and the remaining as “white” (call them \mathcal{W}), and considering all possible “coverings” by singlets (valence bonds) from black to white spins. Given that a spin-1/2 particle has only two orthogonal states, an upper bound for BE in the cooled state, for an initial state with two states, is a reasonable approximate upper bound. Cooling the initial state $|\Psi_0^{\mathcal{H}}\rangle = \prod_{i \in \mathcal{B}} |\psi^{\mathcal{B}}\rangle_i \prod_{i \in \mathcal{W}} |\psi^{\mathcal{W}}\rangle_i$, we obtain $|\Phi_{\text{LR}}^{\mathcal{H}}\rangle$, which is a RVB formed by an equal superposition of all the singlet coverings. Consider a “system” \mathcal{S} of $k \ll 2m$ spins (containing, say, b black spins and w white ones), and the remaining “environment” \mathcal{E} . The orthonormal states of \mathcal{S} in a $\mathcal{S} : \mathcal{E}$ Schmidt decomposition must be fully symmetric under independent permutations in the subsets of black and white spins. So the basis of symmetric spin multiplets $(|b/2, m_b\rangle_{\mathcal{B} \cap \mathcal{S}} |w/2, m_w\rangle_{\mathcal{W} \cap \mathcal{S}})$ will span them, where $m_c = -\frac{c}{2}, \dots, \frac{c}{2}$, ($c = b, w$), so that there are $(b + 1)(w + 1)$ symmetric basis states. So, $E_{k:2m-k} \leq \log_2(b + 1) \times (w + 1)$. No matter what is b or w , any area law of the form $E \sim k^{1-1/D}$ for any D (including $D = \infty$) is out of the question. In the case when $w = k, b = 0$ (or $w = 0, b = k$), the cooled state is

$$\sum_{M=-k/2}^{k/2} |k/2, M\rangle_{\mathcal{B} \cap \mathcal{S}} \sum_{p=-m/2}^{m/2} (-1)^{(p-m/2)} C_{p,M}^{m,k} / \sqrt{m+1} |m - k/2, p - M\rangle_{\mathcal{B} \cap \mathcal{E}} |m/2, -p\rangle_{\mathcal{W}},$$

where $(C_{p,M}^{m,k})^2 = [k, M + k/2][m - k, p - M + (m - k)/2] / [m, p + m/2]$ defines the Clebsch-Gordan coefficient ($C_{p,M}^{m,k} = 0$ unless $0 \leq b \leq a$ in all $[a, b]$ involved), whence one can show that the upper estimate is saturated.

Case 3: 2D $J_1 - J_2 - J_3$ Heisenberg model with RVB ground states.—Consider a (2D) square lattice of size $2m \times 2m$ with periodic boundary conditions (PBC), each site being occupied by a spin-1/2 particle. It is governed by a two-body $J_1 - J_2 - J_3$ AFM Heisenberg Hamiltonian, with exchange constants J_1 for nearest neighbors (NN) and on diagonals in the plaquettes, J_2 for interplaquette NNs and diagonals, and J_3 for next-nearest neighbors (NNN) and knight’s-move-distance-away spins on horizontal and vertical ladders formed by the plaquettes [14,15]. A plaquette is a square of four neighboring spins. Realization of such a “quadrumerized” lattice is being hotly pursued in current experiments with atoms in optical lattices (see [16]). For certain choices of the relative strengths of the couplings, it is reasonable to assume [14,15] that the GS

configuration is formed by either two horizontal singlets $|HH\rangle$ or two vertical ones $|VV\rangle$ in each of the plaquettes, with a fixed density, $d = s/m^2$, of s vertical singlets in the whole state of m^2 plaquettes. A superposition of such states is an example of a RVB state, whereas ordered configurations of dimers (singlets) correspond to valence bond solids (Peierls order). If d is not too close to 0 or 1, the dimension of the GS subspace grows exponentially with system size: for $d = 1/2$, the degeneracy scales as 2^{m^2} . In contrast to cases 1 and 2, the model has rather “medium-ranged” interactions. \mathcal{F} is correspondingly smaller, and e.g. for $J_2 = J_3 = 0$, $\mathcal{F} = 1/2$.

Consider first the scaling $E_{p;\text{rest}}$ for p plaquettes. For the initial state $|\Psi_0^H\rangle = \prod_{p=1}^{m^2} (\alpha|HH\rangle + \beta|VV\rangle)_p$ (unnormalized, and $\alpha\beta \neq 0$), the cooled state (which is independent of α, β) is

$$|\Phi_{\text{RVB}}^d\rangle = \sum_{l=0}^{\min(s,k)} \sum_{r=0}^{\min(s-l, m^2-k)} |l\rangle_{\mathcal{S}} \sqrt{3^l [k, l]} |r\rangle_{\mathcal{E}} \\ \times \sqrt{3^r [m^2 - k, r]} [m^2 - l - r, s - l - r],$$

where $|l\rangle_{\mathcal{S}}$ ($|r\rangle_{\mathcal{E}}$) are orthonormal states of a “system” \mathcal{S} (“environment” \mathcal{E}) of k ($m^2 - k$) plaquettes. When $m^2 \rightarrow \infty$, the upper limit $\min(s-l, m^2-k)$ is l independent, and by Stirling approximation, $\ln[m^2 - t, s - t] \approx (m^2 - t) \times \ln(m^2 - t) - (s - t) \ln(s - t)$ plus functions of m^2 and s , where $t = l + r$. Linearizing the logarithm around the mean \bar{t} of t (in $|\psi_{\text{RVB}}^d\rangle$), and exponentiating back, we obtain a product $[m^2 - t, s - t] \approx q^l q^r$, with $q = (s - \bar{t}) / (m^2 - \bar{t}) \approx [-1 + \sqrt{1 + 12d(1-d)}] / [6(1-d)]$. The approximations are systematic, as $\bar{t} \leq \sqrt{3}k/2$, and dispersion of $t \sim \sqrt{\bar{t}}$. So the states of \mathcal{E} and \mathcal{S} can be well approximated as product states, which further implies that $|\Phi_{\text{RVB}}^d\rangle$, for any d , can be approximated by product over plaquettes, where each plaquette is in the same pure state—a “mean-field picture” is valid, for large m^2 . Consequently, the entanglement of a system \mathcal{S} of size k , plus a boundary \mathbb{B} that intersects Δ plaquettes, to the rest of the lattice, will scale as $hE_{\text{pl}}(d) + vE_{\text{pl}}(1-d)$, where the boundary intersects h (v) plaquettes horizontally (vertically), $h + v = \Delta$, and $E_{\text{pl}}(d) = \log_2(1 + 3q^2) - \frac{3q^2}{1+3q^2} \log_2 q^2$.

For the most general initial state, $|\Psi_0^{H_s}\rangle = \prod_{p=1}^{m^2} (|\psi_1\rangle|\psi_2\rangle|\psi_3\rangle|\psi_4\rangle)_p$, denoting $P_{\square,p}^{S=0}$ for the projector onto the space spanned by $\{|HH\rangle, |VV\rangle\}$ at the p th plaquette, and $\{|G_i\rangle\}$ for the ground states, the cooled state is $(\sum_i |G_i\rangle\langle G_i|) |\Psi_0^{H_s}\rangle = (\sum_i |G_i\rangle\langle G_i|) \prod_{p=1}^{m^2} P_{\square,p}^{S=0} |\Psi_0^{H_s}\rangle = (\sum_i |G_i\rangle\langle G_i|) |\Psi_0^H\rangle$, so that the same area law holds. Surprisingly, therefore, the area law depends on the *path* of the boundary (and not only on its length): it depends on h , even for fixed Δ . However, the area law is linear in this 2D system, and supports our thesis for weakly frustrated systems.

Case 4: Shastry-Sutherland model.—It is a 2D NN antiferromagnet (with Heisenberg couplings of strength J_1) on a square lattice (i, j) , with additional Heisenberg interac-

tions of strength J_2 on the diagonals $(2i, 2j) \leftrightarrow (2i+1, 2j+1)$ and $(2i, 2j+1) \leftrightarrow (2i-1, 2j+2)$, with PBC. The ground state of this model, for $J_1/J_2 < 1/2$, is a product of dimers along the J_2 diagonals [10]. For $J_1/J_2 < 1/2$, by going to the Ising limit, the degeneracy becomes exponentially large, and $\mathcal{F} \approx 1/(1 + (1/2)(J_2/J_1)) < 1/2$, for large system size. Furthermore, the area law is linear, and depends on the path of the boundary.

Case 5: 1D $J_1 - J_2$ Heisenberg model.—Consider a 1D system with PBC and $2m$ sites, having NN and NNN Heisenberg couplings of strengths J_1 and J_2 , respectively. For $J_1 = 2J_2 > 0$ (MG model [11]), the GSs are $|G_{\text{MG}}^\pm\rangle = \prod_{i=1}^m (|0\rangle_{2i}|1\rangle_{2i\pm 1} - |1\rangle_{2i}|0\rangle_{2i\pm 1})/\sqrt{2}$. At the MG point, $\mathcal{F} = 1/2$. Note however that if we replace $\vec{\sigma}_i$ by a 3D classical vector, the frustration degree is zero. Using the initial state $\prod_{i=1}^{m-1} |0\rangle_{2i-1}|1\rangle_{2i}|\phi\rangle_{2m-1}|\phi\rangle_{2m}$, where $|\phi\rangle$ is an arbitrary (qubit) state, we obtain a *lower bound*: $E_{k;\text{rest}} \geq 2$ or 1, for even or odd k . Any cooled state is of the form $|\Phi_{\text{MG}}\rangle = a|G_{\text{MG}}^+\rangle + b|G_{\text{MG}}^-\rangle$, whence, after writing in Schmidt decomposition, an upper bound reads $E_{k;\text{rest}} \leq \log_2 5$ or $\log_2 3$, for even or odd k . Numerical simulations at the MG point, show that, e.g., for $2m = 8$, $E_{4;\text{rest}} \approx 2.314$ (cf. [17]), and BE after cooling converges with k . Therefore, the area law is a constant for this 1D system, in support of our thesis. Note that the above method of finding bounds can potentially be used in other models whose GS space is made up of dimers.

Case 6: Ising spin chain with NN interactions.—The Hamiltonian, of $2m$ spins (with PBC), is $H_{\text{NN}}^{\text{Ising}} = \sum_{\langle ij \rangle} J_{ij} \sigma_i^z \sigma_j^z$, with all $|J_{ij}| = J$, and all except one are negative. $\mathcal{F} = 1/(2m-1)$. The initial state $|\Psi_0^{\text{NN}}\rangle = \prod_i [(|0\rangle + |1\rangle)/\sqrt{2}]_i$ (cf. [18]) gives the cooled state $|\Phi_{\text{NN}}^{\text{Ising}}\rangle = \frac{1}{2\sqrt{m}} \sum_{k=0}^m [(|0^{\otimes(2m-k)} 1^{\otimes k}\rangle + |1^{\otimes(2m-k)} 0^{\otimes k}\rangle) + (|1^{\otimes(k+1)} 0^{\otimes(2m-k-1)}\rangle + |0^{\otimes(k+1)} 1^{\otimes(2m-k-1)}\rangle)]$. For any k , $E_{k;2m-k}$ decreases with $2m$, contrary to previous cases. Moreover, $E_{k;2m-k}$, for a fixed $2m$, is a constant with k —in support of our thesis.

Constructive and destructive interferences.—The above studies allow us to identify an interesting interplay between frustration and interference of entanglement. Beginning with case 3, for a fixed density d , the lattice is in the state $|\psi_{\text{RVB}}^d\rangle$, which is an equal superposition over all states that have s vertical plaquettes and $m^2 - s$ horizontal ones. If we choose one of the states in this superposition, calculate the entropy of the inner area ($\mathcal{S} \cup \mathbb{B}$) in the chosen state, and then average this entropy over all states in the superposition, the average entropy is $\bar{E}(d) = 2hd + 2v(1-d)$. For a square \mathbb{B} , the interference is destructive (i.e., $E/\bar{E} < 1$) for low and high densities, but otherwise constructive, and for a purely horizontal (or vertical) \mathbb{B} , interference can wipe out the entanglement completely, at low densities (Fig. 1). For a square \mathbb{B} , E/\bar{E} is maximal for $d = 1/2$. Correspondingly, in the AFM LR Ising gas, the highest BE scaling is for $\lambda = 0$, when there are an equal number $|0\rangle$'s and $|1\rangle$'s in the components of the cooled state, and \mathcal{F} is

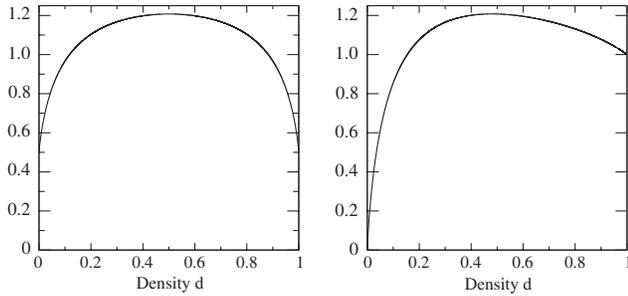


FIG. 1. Constructive and destructive interference. Left: E/\bar{E} for a square \mathbb{B} , so that $v = h$. Right: E/\bar{E} for a horizontal (vertical) boundary, so that $v = 0$ ($h = 0$). The vertical axes are of E/\bar{E} .

also maximal exactly at $\lambda = 0$. We predict that a parent Hamiltonian that describes the ground states in case 3 for fixed density will be maximally frustrated at $d = 1/2$.

For a similar study in the AFMLR Ising gas (case 1), we normalize the entanglement scaling in the frustrated system, i.e., $J > 0$, with the one in the NF one ($J < 0$). Since the latter is unity, the normalized entanglement, to leading order, is $\frac{1}{2} \log_2 k$, for large blocks of size k , large system size $2m$ ($\gg k$), and for all λ . Thus we obtain constructive interference of BE for all λ , with respect to the NF situation. For the MG model, let us once again choose any one of the dimer ground states, find the entanglement in a bipartite split, and average over the two ground states. The result is unity, for any split, so that we again have constructive interference of entanglement. For the system described in case 6, we have marginally constructive interference, compared to the NF case (all $J_{ij} < 0$).

In case 2, we have a rather remarkable example of destructive interference, because when $2m \gg k$, $E_{k,\text{rest}} \sim k$ for most elements in the superposition forming the cooled state, while the latter has only logarithmic scaling at most. Superposition can therefore give rise to qualitative changes in scaling, that are exponentially different.

Summary.—First, our studies show that the dimension of the GS manifold does not provide a “good” characterization of frustration. This can be seen by comparing the $J_1 - J_2$ model at the MG point with the 1D Ising model with a single frustrated bond. Second, we found that trying to confer an “area law” on a frustrated system can have surprising consequences, such as logarithmic divergence of BE in an effectively infinite dimensional system, and dependence of BE on the shape (and not only the Euclidean area) of the boundary. Interestingly, the seeming independence of area law on dimension, in the long-range Ising model, gives rise to the possibility of applying density matrix renormalization group techniques to complex systems with long-range interactions (cf. [19]). Our results indicate that while weakly frustrated systems follow the usual area law known in the literature, strongly frustrated systems will each have their own area law. Finally, we have introduced a cooling procedure to study BE in degenerate

ground state manifolds, a frustration degree, and a method to quantify constructive and destructive interference of entanglement.

We acknowledge comments of H.-U. Everts, R. Fazio, T. Vekua, and support of research project of Polish Government scientific funds (2005-08), ESF QUEDDIS, Marie Curie ATK project COCOS (MTKD-CT-2004-517186), Spanish MEC (FIS-2005-04627/01368, Consolider Project QOIT, and Ramón y Cajal), and EU IP SCALA.

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