Entanglement and quantum groups

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We describe quantum mechanical entanglement in terms of compact quantum groups. We prove an analog of positivity of partial transpose criterion and formulate a Horodecki-type theorem. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Quantum entanglement has been one of the most challenging problems of modern quantum mechanics. We briefly recall here the definition, referring the reader to Ref. 1 for a complete overview. We consider a composite quantum system, composed of two subsystems. If the individual systems are described by Hilbert spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, then, according to the postulates of quantum theory, the composite system is described by the tensor product $\mathcal{H}\otimes\widetilde{\mathcal{H}}$. In this work we will be interested only in those cases when $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ can be chosen to be finite dimensional, e.g., for a pair of spins. A state of the system is represented by a density matrix ϱ, that is, a positive operator from $\mathcal{L}(\mathcal{H}\otimes\widetilde{\mathcal{H}})$, normalized by tr $\varrho=1$. In an obvious way, if ϱ and $\widetilde{\varrho}$ are states of the individual subsystems, then the product $\varrho\otimes\widetilde{\varrho}$ is a state of the compound system. So are convex combinations of such products,

$$\sum_{\lambda} p_{\lambda}\varrho_{\lambda}\otimes\widetilde{\varrho}_{\lambda}, \quad p_{\lambda}\geq 0. \tag{1}$$

It turns out, however, that not every state of the whole system can be represented in the above form[2]—the state space of the composite system is strictly larger. Those states which admit the above representations are called *separable* or *classically correlated* and those which do not—*entangled*. Intuitively, entangled states reflect a very strong correlation between the subsystems. So strong that it even violates a certain locality principle.[3] From a more practical point of view, entanglement is a resource for quantum information processing, e.g., for teleportation, computation, and cryptography.[1] However, the question of an efficient characterization of entangled states turned out to be a very hard task: despite many attempts, this problem still does not possess a satisfactory solution.[1]

In previous works we proposed[4] and developed[5] a novel method of studying (generalized) entanglement using abstract harmonic analysis on ordinary compact groups. The core of the method is the identification of the Hilbert spaces describing the system with representation spaces of some compact groups. Then with the help of Fourier transform, we switch from density matri-

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ces to continuous positive definite functions on direct product of the groups and define and study entanglement in terms of those functions. The main result of that approach[5] is a Horodecki-type theorem,[6] characterizing entanglement of positive definite functions in terms of positive definiteness preserving maps of continuous functions.

In this note we show how the above classical analysis can be extended to compact quantum groups (CQGs).

II. CQGS AND THEIR DIRECT PRODUCTS

We begin with the notation and briefly recall some basic facts. We follow the approach of Woronowicz.[7–9] Let (A,Δ_A) and (B,Δ_B) be two CQGs, where A,B are unital C^*-algebras and Δ_A,Δ_B are the coproducts. Let $\{u^\alpha\}$ and $\{v^\beta\}$ be the complete families of irreducible unitary corepresentations of (A,Δ_A) and (B,Δ_B), respectively. Just like in the classical theory, all such corepresentations are finite dimensional.[8] We denote by $\mathcal{H}_\alpha$ and $\widetilde{\mathcal{H}}_\beta$ the carrier Hilbert spaces of u^α and v^β, respectively, so that u^α and v^β are unitary elements of $\mathcal{L}(\mathcal{H}_\alpha)\otimes A$ and $\mathcal{L}(\widetilde{\mathcal{H}}_\beta)\otimes B$, respectively. Fixing once and forever orthonormal bases $\{e_i^\alpha\}_{i=1,\ldots,n_\alpha}$, $n_\alpha=\dim\mathcal{H}_\alpha$ and $\{\tilde{e}_k^\beta\}_{k=1,\ldots,m_\beta}$, $m_\beta=\dim\widetilde{\mathcal{H}}_\beta$ in each carrier space $\mathcal{H}_\alpha$ and $\widetilde{\mathcal{H}}_\beta$, u^α, v^β can be identified with $n_\alpha\times n_\alpha$ and $m_\beta\times m_\beta$ matrices $[u_{ij}^\alpha]$, $[v_{lk}^\beta]$ with entries in A and B, respectively. They satisfy comultiplication rule, $\Delta_A u_{ij}^\alpha=\Sigma_r u_{ir}^\alpha\otimes u_{rj}^\alpha$, and analogously for $\Delta_B v_{lk}^\beta$.

Let $\mathcal{A}$ ($\mathcal{B}$) be a linear span of all matrix elements u_{ij}^α (v_{kl}^β) of all irreducible corepresentations of (A,Δ_A) $[(B,\Delta_B)]$. This is an analog of the algebra of polynomial functions on an ordinary group. It is a dense $*$-subalgebra of A (B), closed with respect to the comultiplication, and carrying structure of a Hopf algebra.[8,10] We recall (see, e.g., Ref. 8) how counit and coinverse (antipode) maps, defined on the above Hopf algebra, act, respectively, on the matrix elements,

$$\varepsilon_A(u_{ij}^\alpha)=\delta_{ij}, \quad \kappa_A(u_{ij}^\alpha)=u_{ji}^{\alpha *}, \tag{2}$$

and similarly for ε_B,κ_B defined on $\mathcal{B}$.

The main object of our study will be a direct product,

$$(A,\Delta_A)\times(B,\Delta_B)=(A\otimes B,\Delta), \tag{3}$$

$$\Delta := (id\otimes\sigma_{AB}\otimes id)(\Delta_A\otimes\Delta_B), \tag{4}$$

where $\sigma_{AB}:A\otimes B\to B\otimes A$ is the flip operator and the tensor products are the minimal ones. The complete family of unitary irreducible corepresentations $\{U\}$ of $(A\otimes B,\Delta)$ can be chosen in the following form:

$$U_{ikjl}^{\alpha\beta} := u_{ij}^\alpha\otimes v_{kl}^\beta, \tag{5}$$

given by $(n_\alpha m_\beta)\times(n_\alpha m_\beta)$ matrices with entries from $A\otimes B$ (note the labeling). Using definition (4) we check that they satisfy the right comultiplication rule,

$$\Delta U_{ikjl}^{\alpha\beta}=\sum_{r,s} U_{ikrs}^{\alpha\beta}\otimes U_{rsjl}^{\alpha\beta}. \tag{6}$$

The Hopf algebra, associated with the direct product (3) and (4), is given by the algebraic tensor product $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$. The counit and coinverse are naturally defined on it by

$$\varepsilon := \varepsilon_A\otimes\varepsilon_B, \quad \kappa := \kappa_A\otimes\kappa_B. \tag{7}$$

III. QUANTUM FOURIER TRANSFORMS OF DENSITY MATRICES

Let us consider density matrices ϱ on $\mathcal{H}_\alpha\otimes\widetilde{\mathcal{H}}_\beta$, i.e., $\varrho\in\mathcal{L}(\mathcal{H}_\alpha\otimes\widetilde{\mathcal{H}}_\beta)$, $\varrho\geq 0$, tr $\varrho=1$. We perform the following transform (cf. Refs. 4 and 5 and see Ref. 11):

$$\varrho \mapsto \hat{\varrho} := \sum_{i,\ldots,l} \varrho_{ikjl} U^{\alpha\beta}_{jlik} = (\mathrm{tr} \otimes id)\varrho U^{\alpha\beta}, \tag{8}$$

which associates with ϱ an element of $\mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B}$. The indices i,j refer here to the Hilbert space $\mathcal{H}_\alpha$, while k,l to $\widetilde{\mathcal{H}}_\beta$ [cf. definition (5)]. Before describing entanglement, we show how positivity and normalization of ϱ are encoded in $\hat{\varrho}$.

Since, by definition [cf. Eqs. (2) and (7)], $\varepsilon(U^{\alpha\beta}_{ikjl}) = \varepsilon_A(u^\alpha_{ij})\varepsilon_B(v^\beta_{kl}) = \delta_{ij}\delta_{kl}$ we obtain that

$$\varepsilon(\hat{\varrho}) = \mathrm{tr}\ \varrho = 1. \tag{9}$$

To describe the positivity property, let $h = h_A \otimes h_B$ be a unique Haar measure on the product $(A\otimes B, \Delta)$.[7] For convenience we define the following functionals on $A\otimes B$: $ah(b) := h(ba)$, $ha(b) := h(ab)$, $a,b \in A\otimes B$. Then $\hat{\varrho}$ satisfies an analog of positive definiteness,

$$(a^* h\kappa \otimes ha)\Delta\hat{\varrho} \geq 0 \quad \text{for any} \quad a \in A\otimes B. \tag{10}$$

Note that from definitions (2), (5), and (7), $\kappa(U^{\alpha\beta}_{ikjl}) = U^{\alpha\beta *}_{jlik}$.

To prove statement (10), we first use Eq. (6) and then the above quoted property of the coinverse,

$$\begin{aligned}(a^* h\kappa \otimes ha)\Delta\hat{\varrho} &= \sum_{i,\ldots,l} \varrho_{ikjl} \sum_{r,s} h(\kappa(U^{\alpha\beta}_{jlrs})a^*)h(aU^{\alpha\beta}_{rsik}) = \sum_{r,s}\sum_{i,\ldots,l} \varrho_{ikjl} h(U^{\alpha\beta *}_{rsjl} a^*)h(aU^{\alpha\beta}_{rsik}) \\ &= \sum_{r,s}\sum_{i,\ldots,l} \varrho_{ikjl}\overline{h(aU^{\alpha\beta}_{rsjl})}h(aU^{\alpha\beta}_{rsik}) \equiv \sum_{r,s}\langle \psi_{rs}|\varrho\psi_{rs}\rangle \geq 0,\end{aligned} \tag{11}$$

where $\psi_{rs} := \Sigma_{i,k}\overline{h(aU^{\alpha\beta}_{rsik})e^\alpha_i} \otimes \tilde{e}^\beta_k \in \mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta$, with the overbar denoting complex conjugate. We used the identity $h(a^*) = \overline{h(a)}$ and by $\langle\cdot|\cdot\rangle$ we denote the standard scalar product in the corresponding Hilbert spaces.

The main weakness of the proposed definition of positive definiteness (10) is that it can be formulated only on the respective Hopf algebras, since generally coinverse cannot be prolonged to the whole of the quantum group. One possible way out is to take norm closure in $A\otimes B$ of the set of positive definite elements. However, for the purpose of this note, we will not consider such a closure and continue with purely algebraic considerations.

Consider a separable $\varrho \in \mathcal{L}(\mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta)$, i.e., a density matrix ϱ representable as the following finite convex combination:

$$\varrho = \sum_\lambda p_\lambda \varrho^\alpha_\lambda \otimes \varrho^\beta_\lambda, \quad \varrho^\alpha_\lambda, \varrho^\beta_\lambda \geq 0. \tag{12}$$

Then its transform $\hat{\varrho}$ is also *separable*, that is,[12]

$$\hat{\varrho} = \sum_\lambda p_\lambda \hat{\varrho}^\alpha_\lambda \otimes \hat{\varrho}^\beta_\lambda, \tag{13}$$

where for all λ, $\hat{\varrho}^\alpha_\lambda \in \mathcal{A}$ and $\hat{\varrho}^\beta_\lambda \in \mathcal{B}$ satisfy the normalization (9) and the positive definiteness (10) conditions on (A,Δ_A) and (B,Δ_B), respectively.

We would like to have the converse of the above fact. To that end we construct an inverse (in some sense to be clarified later) of the transformation (8). Let F^α, respectively, $\widetilde{F}^\beta$, intertwine the second contragredient representation $u^{\alpha cc} := (id\otimes\kappa_A^2)u^\alpha$ with u^α, respectively, $v^{\beta cc}$ with v^β,[7,10]

$$(id\otimes\kappa_A^2)u^\alpha = (F^\alpha \otimes id)u^\alpha((F^\alpha)^{-1}\otimes id), \tag{14}$$

$$(id \otimes \kappa_B^2)v^\beta = (\tilde{F}^\beta \otimes id)v^\beta((\tilde{F}^\beta)^{-1} \otimes id). \tag{15}$$

Operators F^α, $\tilde{F}^\beta$ are invertible, positive, and uniquely fixed by the condition: $\mathrm{tr}\, F^\alpha = \mathrm{tr}(F^\alpha)^{-1} > 0$, and analogously for $\tilde{F}^\beta$.[8,10] Then $F^{\alpha\beta} := F^\alpha \otimes \tilde{F}^\beta > 0$ intertwines $(id \otimes \kappa^2)U^{\alpha\beta}$ with $U^{\alpha\beta}$. Now, given an arbitrary $a \in A \otimes B$ we define for each α, β an operator $\hat{a}(\alpha\beta)$ on $\mathcal{H}_\alpha \otimes \tilde{\mathcal{H}}_\beta$ by[8,9]

$$\hat{a}(\alpha\beta) := (id \otimes ah)\frac{1}{\sqrt{F^{\alpha\beta}}}U^{\alpha\beta\dagger}\sqrt{F^{\alpha\beta}},$$

$$\hat{a}(\alpha\beta)_{ikjl} = \sum_{m,\ldots,s} (F^{\alpha\beta})^{-1/2}_{ikmn} h(U^{\alpha\beta *}_{rsmn} a)(F^{\alpha\beta})^{1/2}_{rsjl}. \tag{16}$$

Since $U^{\alpha\beta}$ is irreducible, transformation (16) is onto. Recall that matrix element of irreps satisfy deformed orthonormality relations[7,10]

$$h_A(u^{\alpha *}_{ij} u^{\alpha'}_{i'j'}) = \frac{\delta^{\alpha\alpha'}}{\mathrm{tr}\, F_\alpha}(F^\alpha)^{-1}_{i'i}\delta_{jj'} \tag{17}$$

$$h_A(u^{\alpha}_{ij} u^{\alpha' *}_{i'j'}) = \frac{\delta^{\alpha\alpha'}}{\mathrm{tr}\, F_\alpha}\delta_{ii'}F^{\alpha}_{j'j}, \tag{18}$$

and analogously for the irreps of (B, Δ_B). Thus,

$$\varrho = (\mathrm{tr}\, F^{\alpha\beta})\sqrt{F^{\alpha\beta}}\hat{\varrho}(\alpha\beta)\sqrt{F^{\alpha\beta}}, \tag{19}$$

and one can recover ϱ from its transform $\hat{\varrho}$.

Now let a belong to $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}$. We show that if a is *positive definite*, i.e., satisfies [cf. Eq. (10)]

$$(b^* h\kappa \otimes hb)\Delta a \geq 0 \tag{20}$$

for any b from $A \otimes B$, then $\hat{a}(\alpha\beta) \geq 0$ for all α, β. To better understand the condition (20), recall that for an ordinary compact group G, when the relevant C^*-algebra is just the standard C^*-algebra of continuous functions on G, $h(f) = \int dg f(g)$, dg being the Haar measure on G, $(\kappa f)(g) = f(g^{-1})$, $f^*(g) = \overline{f(g)}$, and $(\Delta f)(g,h) = f(gh)$, Eq. (20) is just the standard positive definiteness condition: $\int\int dg dh \overline{b(g)} b(h) a(g^{-1}h) \geq 0$ (see Refs. 4 and 5 for the Fourier analysis of density matrices and separability on ordinary compact groups).

For more transparency, let us introduce compound indices $\boldsymbol{i} \equiv (ik)$ pertaining to $\mathcal{H}_\alpha \otimes \tilde{\mathcal{H}}_\beta$. Since h is normalized, $h(I) = 1$, and $h(\kappa(a)) = a$, we can rewrite $h(U^{\alpha\beta *}_{\boldsymbol{ji}} a) \equiv h(U^{\alpha\beta *}_{jlik} a)$ as

$$\begin{aligned} h(U^{\alpha\beta *}_{\boldsymbol{ji}} a)h(\kappa(I)) &= h\kappa[h(U^{\alpha\beta *}_{\boldsymbol{ji}} a)I] = (h\kappa \otimes h)\Delta(U^{\alpha\beta *}_{\boldsymbol{ji}} a) = (h\kappa \otimes h)\Delta U^{\alpha\beta *}_{\boldsymbol{ji}}\Delta a \\ &= \sum_{\boldsymbol{r}} [\kappa^2(U^{\alpha\beta}_{\boldsymbol{rj}})h\kappa \otimes hU^{\alpha\beta *}_{\boldsymbol{ri}}]\Delta a = \sum_{\boldsymbol{r},\boldsymbol{m},\boldsymbol{n}} F^{\alpha\beta}_{\boldsymbol{rm}}(F^{\alpha\beta})^{-1}_{\boldsymbol{nj}}[U^{\alpha\beta}_{\boldsymbol{mn}}h\kappa \otimes hU^{\alpha\beta *}_{\boldsymbol{ri}}]\Delta a, \end{aligned} \tag{21}$$

where in the second line we used the invariance of the Haar measure: $(h \otimes id)\Delta a = h(a)I = (id \otimes h)\Delta a$, and then Eq. (6) and the identities: $\kappa(ab) = \kappa(b)\kappa(a)$, $U^{\alpha\beta *}_{jlik} = \kappa(U^{\alpha\beta}_{ikjl})$, and finally Eqs. (14) and (15). Substituting Eq. (21) into the definition (16), we obtain

$$\hat{a}(\alpha\beta)_{ij} = \sum (F^{\alpha\beta})^{-1/2}_{im}(F^{\alpha\beta})^{1/2}_{rj}(F^{\alpha\beta})^{-1}_{nr}F^{\alpha\beta}_{r'm'}[U^{\alpha\beta}_{m'n}h\kappa \otimes hU^{\alpha\beta *}_{r'm}]\Delta a$$
$$= \sum [(F^{\alpha\beta})^{1/2}_{sm'}U^{\alpha\beta}_{m'n}(F^{\alpha\beta})^{-1/2}_{nj}h\kappa \otimes h(F^{\alpha\beta})^{1/2}_{r's}U^{\alpha\beta *}_{r'm}(F^{\alpha\beta})^{-1/2}_{im}]\Delta a \equiv \sum_s [U^F_{sj}h\kappa \otimes hU^{F*}_{si}]\Delta a, \tag{22}$$

where $U^F := \sqrt{F^{\alpha\beta}}U^{\alpha\beta}(1/\sqrt{F^{\alpha\beta}})$, and we used Hermiticity of $F^{\alpha\beta}$. Hence, for an arbitrary vector $\psi \in \mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta$,

$$\langle\psi|\hat{a}(\alpha\beta)\psi\rangle = \sum_s \left[\sum_j \psi_j U^F_{sj}h\kappa \otimes h\sum_i \bar{\psi}_i U^{F*}_{si}\right]\Delta a \equiv \sum_s (b^*_s h\kappa \otimes hb_s)\Delta a \geq 0 \tag{23}$$

by positive definiteness of a [cf. Eq. (20)].

Combining the above facts, given by Eqs. (10), (19), and (23), we obtain the following.

Proposition 1: *An operator ϱ acting in the carrier space $\mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta$ is positive if and only if its transform $\hat{\varrho}$ is positive definite, i.e., satisfies condition (10).*

The $\hat{}$-dual of the above Proposition also holds.

Proposition 2: *An element a of the associated Hopf algebra $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$ is positive definite if and only if $\hat{a}(\alpha\beta) \geq 0$ for every irrep α, β.*

The proof in one direction readily follows from Eq. (23). To prove in the other, observe that, by definition, $a \in \mathcal{A}\otimes_{\text{alg}}\mathcal{B}$ is a finite linear combination of matrix elements $U^{\alpha\beta}_{ikjl}$: $a = \sum_{\alpha,\beta}\sum_{i,j} a^{\alpha\beta}_{ij} U^{\alpha\beta}_{ji}$. Then,

$$(b^* h\kappa \otimes hb)\Delta a = \sum_{\alpha\beta}\sum_{i,j,k} a^{\alpha\beta}_{ij} h(bU^{\alpha\beta}_{ki})\overline{h(bU^{\alpha\beta}_{kj})}. \tag{24}$$

If all the matrices $a^{\alpha\beta}_{ij}$ are positive definite, then the above sums are positive and hence a is positive definite. But by Eq. (19) $a^{\alpha\beta} = (\text{tr } F^{\alpha\beta})\sqrt{F^{\alpha\beta}}\hat{a}(\alpha\beta)\sqrt{F^{\alpha\beta}}$. □

IV. SEPARABILITY

In an obvious way the notion of separability given by Eq. (13) applies to an arbitrary element of $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$. The same remark applies here as to the positive definiteness: Since coinverse κ has generally no extension to the whole algebra $A \otimes B$, a way of extending separability to the whole group would be to consider a norm closure in $A \otimes B$ of the set of separable states. Again, we will not pursue this line here and will be satisfied with purely algebraic facts.

We are ready to prove the following fact, justifying the use of CQGs in the study of entanglement (see Refs. 4 and 5 for a classical analog).

Proposition 3: *A density matrix $\varrho \in \mathcal{L}(\mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta)$ is separable if and only if its transform $\hat{\varrho}$ is separable in $\mathcal{A}\otimes_{\text{alg}}\mathcal{B}$.*

The implication in one direction we have already shown [cf. Eq. (13)]. Now assume that $\hat{\varrho}$ is separable,

$$\hat{\varrho} = \sum_\lambda p_\lambda a_\lambda \otimes b_\lambda, \tag{25}$$

where a_λ, b_λ are positive definite for each λ. Then $\hat{\hat{\varrho}}$ is separable too, which immediately follows from Eqs. (5) and (16), and Proposition 1 applied to a_λ, b_λ,

$$\hat{\hat{\varrho}}(\alpha\beta) = \sum_\lambda p_\lambda \hat{a}_\lambda(\alpha) \otimes \hat{b}_\lambda(\beta), \quad \hat{a}_\lambda(\alpha), \ \hat{b}_\lambda(\beta) \geq 0. \tag{26}$$

By Eq. (19),

$$\varrho = (\mathrm{tr}\, F^{\alpha}\, \mathrm{tr}\, \tilde{F}^{\beta})\sqrt{F^{\alpha}} \otimes \sqrt{\tilde{F}^{\beta}}\hat{\varrho}\sqrt{F^{\alpha}} \otimes \sqrt{\tilde{F}^{\beta}}, \tag{27}$$

so ϱ is separable. □

Proposition 3 is a nice theoretical separability criterion that allows to conclude about separability properties of ϱ is the separability properties of $\hat{\varrho}$ are known, and vice versa. So far, we have been, in fact, mainly carrying over the results known for quantum mechanical states to their transforms, either on compact (standard[5] or quantum) groups. Still, having an explicit separable form of $\hat{\varrho}$ implies immediate separability of ϱ.

Using the same technique, combined with the fact that $\mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B} \ni a = \Sigma_{\alpha,\beta}^{\mathrm{finite}}\Sigma_{\boldsymbol{i},\boldsymbol{j}} a_{\boldsymbol{ij}}^{\alpha\beta} U_{\boldsymbol{ji}}^{\alpha\beta}$, where $a^{\alpha\beta} = (\mathrm{tr}\, F^{\alpha\beta})\sqrt{F^{\alpha\beta}}\hat{a}(\alpha\beta)\sqrt{F^{\alpha\beta}}$, we show the following basic.

Proposition 4: *An element $a \in \mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B}$ is separable if and only if the operators $\hat{a}(\alpha\beta)$ are separable for every irrep α, β.*

By Propositions 1 and 4, positive definite (separable) elements of the associated Hopf algebra generate a family of positive (separable) operators, acting in the carrier spaces of irreps of the product $(A \otimes B, \Delta)$. Moreover, each (separable) density matrix can be obtained this way [cf. Eq. (19)]. Thus, in analogy with the classical case,[4,5] a description of separable elements at the level of quantum group would provide a description of separable states in all dimensions, where the given quantum group has irreducible corepresentations. Motivated by this observation we state the following.

Definition 1: (CQG separability problem) *Given a positive definite element of the Hopf algebra $\mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B}$, associated with the group $(A \otimes B, \Delta)$, decide whether it is separable or not.*

Now we derive an analog of positivity of partial transpose (PPT) criterion.[6,13] Note that from the definition (8) it follows that

$$\widehat{\varrho^{T}} = \sum_{\boldsymbol{i},\boldsymbol{j}} \varrho_{\boldsymbol{ij}} U_{\boldsymbol{ij}}^{\alpha\beta} = \sum_{\boldsymbol{i},\boldsymbol{j}} \varrho_{\boldsymbol{ij}} \kappa(U_{\boldsymbol{ji}}^{\alpha\beta})^{*}. \tag{28}$$

This suggests the following definition of a "transposition map" θ:

$$\theta(a) \coloneqq \kappa(a)^{*}. \tag{29}$$

Note that, quite surprisingly, θ is a homomorphism rather than an antihomomorphism of the associated Hopf algebra: $\theta(ab) = \theta(a)\theta(b)$, but, on the other hand, it is antilinear. From Propositions 1 and 2 we immediately obtain the following.

Proposition 5: *Let ϱ* act *in $\mathcal{H}_{\alpha}\otimes\tilde{\mathcal{H}}_{\beta}$. Then $\varrho^{T_2} \geq 0$ if and only if $(id \otimes \theta)\hat{\varrho}$ is positive definite.*

We prove the following analog of the PPT criterion.

Theorem 1: (Quantum PPT criterion) *If an element $a \in \mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B}$ is separable, then $(id \otimes \theta_B)a$, or equivalently $(\theta_A \otimes id)a$, is positive definite.*

First we need the following fact, which we prove in Appendix.

Proposition 6: *If a is positive definite in $\mathcal{A}$ and b is positive definite in $\mathcal{B}$, then $a \otimes b$ is positive definite in $\mathcal{A}\otimes_{\mathrm{alg}}\mathcal{B}$.*

Then it is enough to show that if a is positive definite, then $\theta(a)$ is positive definite as well. Using the notation $\Delta a = \Sigma_{(a)} a_1 \otimes a_2$ one obtains

$$\begin{aligned}(b^{*}h\kappa \otimes hb)\Delta\theta(a) &= \sum_{(a)} (b^{*}h\kappa \otimes hb)\kappa(a_2)^{*} \otimes \kappa(a_1)^{*} = \sum_{(a)} h[\kappa(\kappa(a_2)^{*})b^{*}]h(b\kappa(a_1)^{*}) \\ &= \sum_{(a)} \overline{h(\kappa(a_1)b^{*})h[b\kappa(\kappa(a_2)^{*})^{*}]} = \sum_{(a)} \overline{h(\kappa(a_1)b^{*})h(ba_2)} = \overline{(b^{*}h\kappa \otimes hb)\Delta a} \geq 0,\end{aligned} \tag{30}$$

where we used anticommultiplicativity of κ (cf. Ref. 8, Proposition 1.9): $\Delta\kappa = (\kappa \otimes \kappa)\sigma\Delta$ (σ is the flip) and the identity $\kappa(\kappa(a)^{*})^{*} = a$. □

Finally, we state the following.[5,6]

Theorem 2: (Quantum Horodecki theorem) *An element a of the quantum group $(A \otimes B, \Delta)$ is* *separable if and only if for every bounded linear map* $\Lambda : B \rightarrow A$ *preserving positive definiteness,* $(id \otimes \Lambda)a$ *is positive definite.*

The proof will be given elsewhere.

V. A $SU_q(2)$ EXAMPLE

Here we present a simple example of the transform (8). We will consider a $2 \otimes 2$-dimensional quantum system in the singlet state,

$$\Psi_- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{31}$$

where we use Dirac notation and $|01\rangle$ stands for the product of the basis elements $e_0 \otimes \tilde{e}_1$, etc.

As the underlying groups we choose two copies of the quantum deformation of $SU(2)$, i.e., $SU_q(2)$. Recall[8,14] that $SU_q(2)$ is obtained from a universal $*$-algebra generated by two generators a, c satisfying the relations

$$ac = qca, \quad cc^* = c^*c, \quad ac^* = qc^*a,$$

$$a^*a + \frac{1}{q}c^*c = aa^* + qcc^* = I, \tag{32}$$

where $q \in [-1, 1]$, $q \neq 0$. Note that for $q = 1$ the resulting algebra is commutative and one recovers the standard $SU(2)$ group. Comultiplication is defined by

$$\Delta(a) \coloneqq a \otimes a - c^* \otimes c,$$

$$\Delta(c) \coloneqq a \otimes c + c \otimes a^*, \tag{33}$$

and coinverse by

$$\kappa(a) \coloneqq a^*, \quad \kappa(c) \coloneqq -\frac{1}{q}c,$$

$$\kappa(c^*) \coloneqq -qc^*, \quad \kappa(a^*) \coloneqq a. \tag{34}$$

The fundamental corepresentation is given by the unitary matrix,

$$u = \begin{bmatrix} a & \sqrt{q}c \\ -\frac{1}{\sqrt{q}}c^* & a^* \end{bmatrix}. \tag{35}$$

It is enough to consider the fundamental corepresentation of $SU_q(2) \times SU_q(2)$. Thus, as the irrep U we take $u \otimes u$. Inserting Eqs. (31) and (35) into Eq. (8) (and stretching the notation a bit), we obtain

$$\hat{\Psi}_- = \langle \Psi_- | u \otimes u \Psi_- \rangle = \tfrac{1}{4}(a \otimes a^* + a^* \otimes a + c \otimes c^* + c^* \otimes c). \tag{36}$$

It is now a highly nontrivial fact, which follows from our analysis, that the above element cannot be represented as a convex combination of products of positive definite elements.

VI. CONCLUSIONS

This paper follows the research lines of Refs. 4 and 5 and relates the separability problem in quantum mechanics to abstract problem of separability of positive definite functions on compact groups, and now on CQGs. So far we have mainly "translated" known results from the entanglement theory to harmonic analysis on the corresponding groups. In particular, Proposition 3 of the present paper pointed out equivalence of the separability of the states and their corresponding transforms. We strongly believe that further studies of harmonic analysis, in particular, in the case of finite groups, will allow to obtain novel results concerning the separability problem in quantum mechanics. One of the goals of this series of papers is indeed to stimulate the interest of mathematicians and mathematical physicists working in harmonic analysis in the separability problem.

In the course of our analysis, we have introduced a notion of positive definiteness (20). There are natural notions of positive elements in C^*-algebras as well as positive and completely positive maps of C^*-algebras. For ordinary groups, positive definite functions define positive functionals on the convolution algebra, while positive maps correspond to positive definiteness preserving maps.[5] In this context, note that condition (20) can be rewritten as $((hb)^* * hb)a \geq 0$, where the convolution of linear functionals is defined as[8] $(\eta' * \eta)a := (\eta' \otimes \eta)\Delta a$ and the involution as $\eta^*(a) := \overline{\eta(\kappa(a)^*)}$.

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APPENDIX: PROOF OF PROPOSITION 6

Let a and b be positive definite elements from $\mathcal{A}$ and $\mathcal{B}$, respectively. We use the notation $\Delta a = \Sigma_{(a)} a_1 \otimes a_2$, $\Delta b = \Sigma_{(b)} b_1 \otimes b_2$ For any finite linear combination $c = \Sigma_{i,k} c_{ik} c_i^A \otimes c_k^B$ from $A \otimes B$, it then holds

$$\begin{aligned}(c^* h\kappa \otimes hc)\Delta(a \otimes b) &= \sum_{(a),(b)} (c^* h\kappa \otimes hc) a_1 \otimes b_1 \otimes a_2 \otimes b_2 \\ &= \sum_{(a),(b)} h(\kappa_A(a_1) \otimes \kappa_B(b_1)c^*) h(ca_2 \otimes b_2) \equiv \sum_{i,\ldots,l} \overline{c_{ik}} c_{jl} H_{ij}^A H_{kl}^B, \end{aligned} \tag{A1}$$

where

$$H_{ik}^A := \sum_{(a)} h_A(\kappa_A(a_1) c_i^{A*}) h_A(c_k^A a_2), \tag{A2}$$

and analogously for H_{jl}^B. Since a and b are positive definite, H^A and H^B are positive definite matrices and so is their tensor product $H^A \otimes H^B$. Hence, $\Sigma_{i,\ldots,l} \overline{c_{ik}} c_{jl} H_{ij}^A H_{kl}^B \geq 0$. Since any c from $A \otimes B$ is a norm limit of linear combinations of product elements and Haar measure h is norm continuous, we obtain that $(c^* h\kappa \otimes hc)\Delta(a \otimes b) \geq 0$ for any c. □

1. R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, e-print arXiv:quant-ph/0702225v2.
2. E. Schrödinger, Naturwiss. **23**, 807 (1935).
3. J. S. Bell, *The Theory of Local Beables* (Cambridge University Press, Cambridge, 2004) [reprinted in J. S. Bell, *Speakable and Unspeakable in Qunatum Mechanics* (Cambridge University Press, Cambridge, 2004)].
4. J. K. Korbicz and M. Lewenstein, Phys. Rev. A **74**, 022318 (2006).
5. J. K. Korbicz, J. Wehr, and M. Lewenstein, Commun. Math. Phys. **281**, 753 (2008).
6. M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
7. S. L. Woronowicz, *Compact Quantum Groups* (Warsaw University, Warsaw, 1991).
8. S. L. Woronowicz, Commun. Math. Phys. **111**, 613 (1987).
9. P. M. Sołtan and S. L. Woronowicz, e-print arXiv:0705.2527v2.
10. T. H. Koornwinder, in *Representations of Lie Groups and Quantum Groups*, Pitman Research Notes in Mathematics Series Vol. 311, edited by V. Baldoni and M. A. Picardello (Longmans, New York, 1994).

[11] Obviously, transformation (8) can be performed for any operator from $\mathcal{L}(\mathcal{H}_\alpha \otimes \widetilde{\mathcal{H}}_\beta)$, but we are interested here only in density matrices.
[12] We use a trivial fact: $\widehat{\varrho_A \otimes \varrho_B} = \hat{\varrho}_A \otimes \hat{\varrho}_B$, following from Eq. (5).

[13] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[14] S. L. Woronowicz, Publ. Res. Inst. Math. Sci. **23**, 117 (1987).